An almost-inviscid geostrophic flow

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An almost rigid rotation of a viscous fluid is produced by dividing the containing cylinder into two sections and rotating them at slightly different speeds. The fluid velocity can be separated into two parts, a swirl about the axis and a streaming motion in the axial planes. When the difference in the speeds of rotation of the two sections is small, the equations of motion can be linearized. The solution is found for large Reynolds numbers and provides an illustration of the way in which the conditions of geostrophic flow (no velocity variation in the axial direction and an inability to insist on undisturbed flow at infinity) are approached as the Reynolds number tends to infinity.

A geostrophic flow is a small relative motion imposed on a steady rotation of an inviscid fluid. It is known that the rotation prevents any variation of velocity in the direction of the rotation vector. It is impossible, therefore, to maintain a boundary condition of undisturbed flow at infinity and the motion is in some cases indeterminate. A way of avoiding this difficulty is to solve the corresponding viscous flow problem and then to consider the inviscid limit, but this process is, in general, too difficult to carry out. It is, however, possible to solve the viscous flow problem in certain simple cases, one of which is discussed here to illustrate the way in which the conditions of geostrophic flow are reached as the Reynolds number increases.

The example chosen is a long cylinder divided into two sections rotating with slightly different speeds. The enclosed fluid will rotate rigidly with the cylinder at a large distance from the plane of division and the solution of the equations of motion will show how the speed of rotation varies along the axis and will also give the nature of the circulatory flow produced by the axial variation of the radial pressure gradient associated with the rotation.

If (r, θ, z) are cylindrical polar co-ordinates, the velocity components (u, v, w) are independent of θ. The angular velocity of the cylinder is $\Omega_1 \pm \Omega_2$ according as $z \gtrless 0$, and the radius of the cylinder is a. A non-dimensional stream function is defined by

$$u = -\frac{a\Omega_2}{r}\frac{\partial\psi}{\partial z}, \quad w = \frac{a\Omega_2}{r}\frac{\partial\psi}{\partial r}, \tag{1}$$

and the azimuthal velocity can be written

$$v = a\Omega_1 r + a\Omega_2 \chi/r. \tag{2}$$

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Since only a slight variation in the speed of rotation of the cylinder is postulated, Ω_2/Ω_1 is small and only the first-order terms in ψ and χ in the Navier–Stokes equations need be included, which yields the equations (see Proudman 1956)

$$\left\{\frac{\partial^2}{\partial r^2}-\frac{\partial}{r\,\partial r}+\frac{\partial^2}{\partial z^2}\right\}^2 \psi = 2R\frac{\partial \chi}{\partial z}, \tag{3}$$

$$\left\{\frac{\partial^2}{\partial r^2}-\frac{\partial}{r\,\partial r}+\frac{\partial^2}{\partial z^2}\right\} \chi = -2R\frac{\partial \psi}{\partial z}, \tag{4}$$

where the Reynolds number $R = \Omega_1 a^2/\nu$. All lengths are non-dimensional with a as unit. The boundary conditions on the cylinder are

$$\psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = 1, \tag{5}$$

$$\chi = \pm 1 \quad \text{at} \quad r = 1, z \gtrless 0, \tag{6}$$

and, at infinity,

$$\psi \to 0 \quad \text{as} \quad z \to \pm\infty, \tag{7}$$

$$\chi/r^2 \to \pm 1 \quad \text{as} \quad z \to \pm\infty. \tag{8}$$

The solution of these equations with R infinite is the geostrophic flow and in this case ψ and χ are functions of r only which, as mentioned above, is a well known result for such flows. Since the fluid is now inviscid, the boundary conditions (5) and (6) must be replaced by $\psi = 0$ on $r = 1$ only, and it is clear that the conditions at infinity cannot be satisfied. All that can be said about the geostrophic flow is that it consists of a rotation and a streaming motion parallel to to the axis, both of which depend in an arbitrary way on the distance from the axis. When the solution with R finite is found, however, this arbitrariness disappears.

The equations (3) and (4) give a sixth-order equation for the stream function

$$\left\{\frac{\partial^2}{\partial r^2}-\frac{\partial}{r\,\partial r}+\frac{\partial^2}{\partial z^2}\right\}^3 \psi = -4R^2\frac{\partial^2 \psi}{\partial z^2}, \tag{9}$$

and the solution proportional to $\cos uz$ and finite on the axis is

$$\cos uz\{ArI_1(u_1 r)+BrI_1(u_2 r)+CrI_1(u_3 r)\},$$

where

$$\left.\begin{aligned} u_1^2 &= u^2+(2Ru)^{\frac{2}{3}},\\ u_2^2 &= u^2+\omega(2Ru)^{\frac{2}{3}},\\ u_3^2 &= u^2+\omega^2(2Ru)^{\frac{2}{3}},\end{aligned}\right\} \tag{10}$$

and ω is a complex cube root of unity. The stream function, which is an even function of z, can be written

$$\psi = \int_0^\infty r\{A(u)\,I_1(u_1 r)+B(u)\,I_1(u_2 r)+C(u)\,I_1(u_3 r)\}\cos uz\,du, \tag{11}$$

and the corresponding form for χ, found from (3), is

$$\chi = \int_0^\infty (2Ru)^{\frac{1}{3}}\, r\{A(u)\,I_1(u_1 r)+\omega^2 B(u)\,I_1(u_2 r)+\omega C(u)\,I_1(u_3 r)\}\sin uz\,du. \tag{12}$$

The boundary conditions (5) for ψ and (6) for χ give

$$AI_1(u_1)+BI_1(u_2)+CI_1(u_3) = 0,$$
$$Au_1 I_0(u_1)+Bu_2 I_0(u_2)+Cu_3 I_0(u_3) = 0,$$
$$AI_1(u_1)+B\omega^2 I_1(u_2)+C\omega I_1(u_3) = 2\pi^{-1}(2R)^{-\frac{1}{3}}u^{-\frac{4}{3}}.$$

Solving for the coefficients and substituting in (11) and (12) gives

$$\psi = \frac{2i(2R)^{-\frac{1}{3}}r}{\pi\sqrt{3}}\int_0^\infty \frac{\alpha_1 I_1(u_1 r)+\alpha_2 I_1(u_2 r)+\alpha_3 I_1(u_3 r)}{u^{\frac{4}{3}}\Delta}\cos uz\,du, \tag{13}$$

$$\chi = \frac{2ir}{\pi\sqrt{3}}\int_0^\infty \frac{\alpha_1 I_1(u_1 r)+\omega^2\alpha_2 I_1(u_2 r)+\omega\alpha_3 I_1(u_3 r)}{u\Delta}\sin uz\,du, \tag{14}$$

where
$$\alpha_1 = u_3 I_0(u_3)\,I_1(u_2)-u_2 I_0(u_2)\,I_1(u_3), \tag{15}$$

and α_2 and α_3 are defined similarly by interchanging the suffixes cyclically, and

$$\Delta = u_1 I_0(u_1)\,I_1(u_2)\,I_1(u_3)+\omega^2 u_2 I_0(u_2)\,I_1(u_3)\,I_1(u_1)+\omega u_3 I_0(u_3)\,I_1(u_1)\,I_1(u_2). \tag{16}$$

If it is now supposed that R is large, (10) can be replaced by

$$u_1 = (2Ru)^{\frac{1}{3}}, \quad u_2 = u_1\exp(\pi i/3), \quad u_3 = u_1\exp(-\pi i/3), \tag{17}$$

and the roots of $\Delta = 0$ are at the points $u_1 = \lambda_s \exp\{(2n+1)\pi i/6\}$ (s and n integral), where the first three values of λ_s are

$$\lambda_1 = 4{\cdot}36, \quad \lambda_2 = 7{\cdot}54, \quad \lambda_3 = 10{\cdot}70,$$

and the asymptotic expansion of the Bessel functions yields the approximate expression for the roots,

$$\lambda_s = \frac{5\pi}{12}+s\pi-\frac{3}{8}\left(\frac{5\pi}{12}+s\pi\right)^{-1}. \tag{18}$$

This formula gives even λ_1 to within 1 % of its exact value.

If the substitution $2Ru = \lambda^3$ is made, (13) becomes

$$\psi = \frac{r\sqrt{3}}{\pi}\int_c \frac{i\{\alpha_1 I_1(u_1 r)+\alpha_2 I_1(u_2 r)+\alpha_3 I_1(u_3 r)\}}{\lambda^2\Delta}\exp(i\lambda^3 z/2R)\,d\lambda, \tag{19}$$

where the integral is taken round the sector bounded by the lines $\arg\lambda = 0$ and $\arg\lambda = \frac{1}{3}\pi$, as the integrals along the straight portions are equal and the integral along the circular arc tends to zero as the radius tends to infinity. The poles inside the contour are at the points $\lambda = \lambda_s\exp(\pi i/6)$ and the corresponding values of the u_i are $u_1 = \lambda_s\exp(\pi i/6)$, $u_2 = \lambda_s\exp(\pi i/2)$, $u_3 = \lambda_s\exp(-\pi i/6)$. There is no pole at $\lambda = 0$ as the numerator and denominator are both of order λ^9 there. The determination of the residues at the poles involves some lengthy manipulation. Writing

$$J_0\{\lambda\exp(\pi i/3)\} = a_0(\lambda)+ib_0(\lambda),$$

and
$$J_1\{\lambda\exp(\pi i/3)\} = a_1(\lambda)+ib_1(\lambda),$$

where λ, a_0, a_1, b_0, b_1 are all real, (19) becomes

$$\psi = 2\sqrt{3}r\sum_{s=1}^{\infty}\frac{N_1(\lambda_s,r)\exp(-\lambda_s^3|z|/2R)}{\lambda_s^2 D(\lambda_s)}, \tag{20}$$

where the values of λ_s are given by (18) and N_1 and D are defined by

$$N_1(\lambda, r) = -J_1(\lambda r)\{a_0(\lambda)\,b_1(\lambda) - a_1(\lambda)\,b_0(\lambda) - \sqrt{3}\,a_0(\lambda)\,a_1(\lambda) - \sqrt{3}\,b_0(\lambda)\,b_1(\lambda)\}$$
$$+ a_1(\lambda r)\{2J_0(\lambda)\,b_1(\lambda) - J_1(\lambda)\,b_0(\lambda) - \sqrt{3}\,J_1(\lambda)\,a_0(\lambda)\}$$
$$+ b_1(\lambda r)\{-2J_0(\lambda)\,a_1(\lambda) + J_1(\lambda)\,a_0(\lambda) - \sqrt{3}\,J_1(\lambda)\,b_0(\lambda)\}, \tag{21}$$

$$D(\lambda) = J_1(\lambda)\{3a_1^2(\lambda) + 3b_1^2(\lambda) + a_0^2(\lambda) + b_0^2(\lambda)\}$$
$$- 2J_0(\lambda)\{a_0(\lambda)\,a_1(\lambda) + b_0(\lambda)\,b_1(\lambda)\}. \tag{22}$$

By the same procedure, (14) becomes

$$\chi = r^2 + 2\sqrt{3}\,r \sum_{s=1}^{\infty} \frac{N_2(\lambda_s, r)\exp(-\lambda_s^3 z/2R)}{\lambda_s D(\lambda_s)} \quad (z > 0), \tag{23}$$

where

$$N_2(\lambda, r) = J_1(\lambda r)\{-a_0(\lambda)\,a_1(\lambda) - b_0(\lambda)\,b_1(\lambda) + \sqrt{3}\,a_0(\lambda)\,b_1(\lambda) - \sqrt{3}\,a_1(\lambda)\,b_0(\lambda)\}$$
$$+ a_1(\lambda r)\{-2J_1(\lambda)\,b_0(\lambda) - J_0(\lambda)\,a_1(\lambda) + \sqrt{3}\,J_0(\lambda)\,b_1(\lambda)\}$$
$$+ b_1(\lambda r)\{2J_1(\lambda)\,a_0(\lambda) - J_0(\lambda)\,b_1(\lambda) - \sqrt{3}\,J_0(\lambda)\,a_1(\lambda)\}. \tag{24}$$

The first term in (23) comes from the singularity at the origin. The value of χ for $z < 0$ is given by $\chi(-z) = -\chi(z)$. It is clear that the boundary conditions (7) and (8) at infinity are satisfied by these values of ψ and χ.

The approximation (17) for the correct value (10) of u is valid if the values of λ used in (20) and (23) are small compared with $R^{\frac{1}{2}}$. The exponential factor in these expressions ensures that the larger values of λ only affect them when z is very small, so that the approximation can be made provided z is large compared with $R^{-\frac{1}{2}}$. An alternative argument, suggested by a referee, indicates that the solution is, in fact, valid for all z, as it is the first term in a spatially uniform expansion of the solution of the complete equations in powers of $1/R$. The approximation (17) is equivalent to dropping the z-derivatives from the operator on the left-hand side of (9) and the solution obtained on the basis of this approximation has r-derivatives $O(1)$ and z-derivatives $O(R^{-1})$, which justifies the approximation. In confirmation of the validity for all z of the solution given here, Proudman's (1956) proof of the impossibility of a shear layer in a plane $z =$ constant may be noted.

The stream lines for the circulatory motion are shown in figure 1 and the angular velocity $\omega = \chi/r^2$ in figure 2. Since z and R only occur in the combination z/R in ψ and χ, it is easy to see how the velocities alter as R is increased. The axial velocity is the same at points with the same values of z/R but the radial velocity decreases by a factor $1/R$ so that the flow becomes increasingly close to an axial flow with the same velocity at all values of z, i.e. it approaches a geostrophic flow. The inviscid limit is a rotation with the mean angular velocity Ω_1 and an axial flow with no net transfer of fluid across any plane $z =$ constant, with the direction of flow changing on the cylinder of radius $0{\cdot}6a$, approximately.

A feature of figure 2 which requires explanation is that, except near the wall, the angular velocity, instead of increasing monotonically from $\Omega_1 - \Omega_2$ to $\Omega_1 + \Omega_2$ as z increases from large negative to large positive values, first increases, then decreases and finally increases again. This behaviour appears in figure 2, since the curves, which are for positive z only, lie below the axis for z/R and r sufficiently small. (The second crossing of the axis by the curve for $z/R = 0{\cdot}001$ is probably an error due to the difficulty of calculation at very small values of z/R.) In

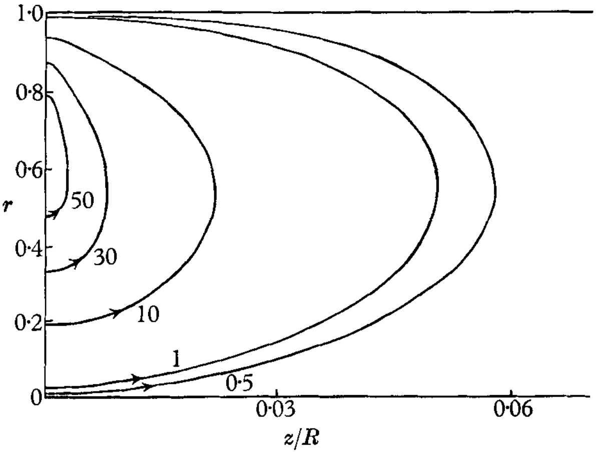

FIGURE 1. The stream lines of the circulatory flow. The numbers on the stream lines are the values of $10^3\psi$ and the flow is in the direction indicated if $z > 0$ is the faster section of the cylinder.

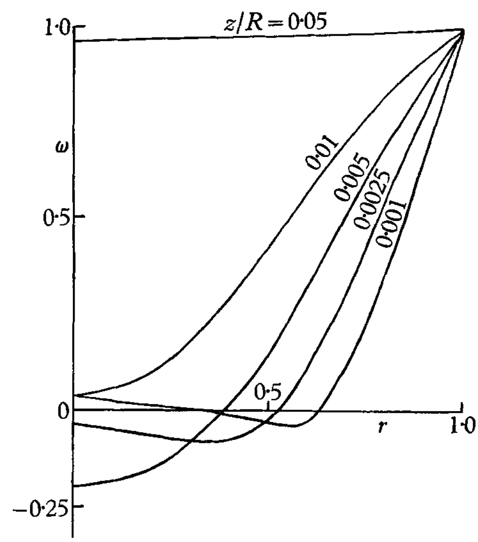

FIGURE 2. The angular velocity relative to the mean rotation.

meteorological terms, the behaviour can be described as a cyclonic flow changing to an anti-cyclonic flow as $z = 0$ is approached through positive values. An explanation can be given in terms of the circulation taken round circles of constant r, which satisfies equation (4),

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\partial}{r\,\partial r} + \frac{\partial^2}{\partial z^2}\right)\chi = -2R\frac{\partial\psi}{\partial z} = -\frac{2\Omega_1 a^2 r^2}{\nu r}\frac{\partial\psi}{r\,\partial z}, \tag{25}$$

where χ is the circulation, or rather the deviation of the circulation from that of the mean rotation of the fluid, $\Omega_1 a^2 r^2$. The left-hand side of the equation represents the effect of viscosity on the circulation and by itself would give a monotonically increasing value for χ. The other part of the equation is the only first-order inertial effect and is the radial convection of the *mean* circulation. This term is only important when the radial velocity is not negligible so that it does not affect the circulation for large z. The direction of the radial velocity is towards the axis for negative z and away from the axis for positive z and the convection is sufficient to make χ positive for some negative values of z and negative for some positive values. A comparison of figures 1 and 2 confirms that the minimum value of χ as a function of z for positive z occurs where the radial velocity is largest.

REFERENCE

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